Enhancing Children's Understanding of Mathematical Concepts Using Pantomime

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Abstract

The purpose of this paper is to show how the use of pantomime enhances children's understanding of mathematical concepts. It is not easy for children to understand most mathematical concepts when they learn mathematics in the classroom. Pantomime is a performance in which a story is told by expressive bodily or facial movements of a performer. Because pantomime is performed without saying anything, children should watch very carefully what a performer's movements mean. In the classroom, when the teacher performs pantomime, the children are supposed to express their own thinking based on the pantomime.

To teach mathematical concepts to young children, the teacher could wear a clown outfit, while performing the pantomime. Pantomime is a powerful tool to teach mathematics because it enhances children's communication, reasoning, and the correct formation of mathematical concepts. For example, it is not easy for students to understand the difference of measurement division and partitive division. The pantomimes of measurement division and partitive division should be different, and children are drawing figures, making formula, putting mathematical terms, and using symbols to represent the situations in the pantomimes. Children have opportunities to exchange their ideas, revise their drawings and writings, and represent their ideas in front of the whole class. They spontaneously reflect on their own ideas and represent various ways to fit the pantomime situations. Thus, pantomime helps children understand mathematical concepts.

For enhancing children's logical and creative thinking, pantomimes can be designed for children to think in various ways. Pantomime provides a wonderful opportunity for children to think of different traditional strategies and to help them build conceptual understanding of mathematical concepts. In addition, through the use of pantomime, children could strengthen logical and creative thinking, thoughtful observation, meaningful representation and effective communication as well as develop a positive attitude toward mathematics.

Enhancing Children's Understanding of Mathematical Concepts Using Pantomime

1. **Why Pantomime?**

Pantomime is a performance in which a performer expresses his or her intention by expressive bodily or facial movements. Because pantomime is performed without saying anything, audiences should watch very carefully what the performer's movements mean. When a teacher performs a pantomime, students may understand various ways about what the pantomime means. Also, students themselves can perform pantomimes to represent or to express mathematical concepts. Students express their own thinking based on their understanding, and their understandings are critically influenced by their experiences. Students naturally respond in various ways because their experiences are various.

To teach mathematics for students, speakers have used pantomimes for many years. Wearing a clown outfit, the speaker, and the teacher, performs pantomimes in which the teacher intends to teach mathematical concepts to the students. Pantomimes are powerful tools to teach mathematics because they enhance students' communication, reasoning, and right formation of mathematical concepts.

For example, it is not easy for students to understand the difference between measurement division and partitive division. The pantomimes of measurement division and partitive division should be different, and students are drawing figures, making formula, putting mathematical terms, and using symbols to represent the situations in the pantomimes. During class, students have opportunities to exchange their ideas, revise their drawings and writings, and present their ideas in front of the whole class. They spontaneously reflect on their own ideas and represent various ways to fit the pantomime situations.

Through pantomime students are able to construct mathematical concepts from the situations relating to students' daily lives. Motivated by pantomime, students spontaneously participate in the mathematics class. Thus, by expressing from pantomime situations into mathematical situations, students can communicate and construct mathematical concepts correctly. In addition, pantomime can enhance students' logical thinking by providing students a step-by-step approach of experiencing to represent everyday situations into mathematical terms and symbols. It also stimulates students' creative thinking by giving them opportunities to draw figures, to name their own terms, and to make their own formula.

2. **An Episode**

If a student asks a question as below, the teacher may answer in two cases.

Student 1: What do you think $3 \div 1/2$ is?
Teacher: I think that the answer of $3 \div 1/2$ is 6.
Student 1: Why do you think $3 \div 1/2 = 6$?
Teacher: How should I answer for that?
Student 2: What do you think $1/2 \div 4$ is?
Teacher: I think that the answer is $1/2 \div 4 = 1/8$.
Student 2: Why do you think $1/2 \div 4 = 1/8$?

Teacher: How should I answer for that?

To answer these questions, students should have the concept of division. Answering like $3 \div 1/2 = 3 \times 2 = 6$ and $1/2 \div 4 = 1/2 \times 1/4 = 1/8$ is not right answer. This response just shows the process of computation, but not the reason of the answer of the problem.

3. **Use of Concept**

One of the most important goals in mathematics education is for students to develop logical and creative thinking. How can we help them to achieve the goal effectively? We often say that "we have to know mathematical concepts to be good at doing mathematic," or "we have to have strong foundations of basics to be good at doing mathematics." The meaning of "have to have strong foundation" is that we have to have correct mathematical basic concepts. That is, well-formed mathematical concepts are needed to be good at doing mathematics. For this reason, pantomime is a powerful method to help students to foster their mathematical understanding.

The term "mathematical concept" is not used in pure mathematics but in mathematics education. We can hardly find the definition of "concept" in the college mathematics textbooks. The term "mathematical concept" is usually used in the field of mathematics education when we are teaching and learning mathematics.

The lexical definition of mathematical concept is "an abstract or generic idea generalized from many particular instances," or "mental construction by removing (eliminating) things that is not in common and pulling out (abstracting) common properties out of many objects, phenomena, and relations."

However, we are embarrassed when we are asked to take a concrete example of the above definition of mathematical concepts. For example, if we are asked, "what is the concept of $6 \div 2 = 3$?" It is not easy to explain the concept of $6 \div 2 = 3$ based on the above definition. As another example, if asked, "what is the improper fraction?" we may answer that "improper fraction is the fraction that is a numerator is equal to or greater than the denominator such as 4/4 or 5/4. "

However, this explanation is not the definition according to the above lexical definition. The examples show that the explanation is not for the concept of improper fraction, but for the terms themselves. For instance, this kind of approach of the meaning of the term "improper fraction" is not "some parts out of a whole," but "a fake fraction" or "inappropriate fraction." The Korean and Chinese terms of improper fraction 가분수(假分數), which means 가假(fake) + 분수分數 are (fraction), and English meaning of improper is "im (not)" + "proper (suitable)." Thus, the term itself is not enough to represent the mathematical concept of improper fraction.

I do not mean that the lexical definition of concept is wrong, but my interest is that how we as teachers can help students to develop correct mathematical concepts. As an alternative way of enhancing this, I will introduce pantomime.

4. Definition of Concept and Mathematical Concept

Even if we use the term "concept" in our daily lives, it is not easy to define clearly the meaning of it. We usually use the term if we can guess the indicated thing when other person points a thing. In this case, we can usually say that we have the concept of it. For example, we can say that we have some sorts of the concept of "a jungle gym" if we can imagine that the signified jungle gym because we had experiences of playing in, seeing, or reading about a jungle gym. In our daily lives, concept and knowledge are used interchangeably. The lexical definition mentioned above, "an abstract or generic idea generalized from particular instances" is not enough to define the meaning of "concept."

Richard Skemp also said that it is not easy to define "concept" and defined "concept" as mental representation of common characteristics out of many objects. Skemp classified concepts into various kinds such as first-order concept and second-order concept with the degree of sensory experience. Also, he differentiated primary and secondary concepts according to whether they are derived directly from sensory experiences or they depend on other concepts, respectively.

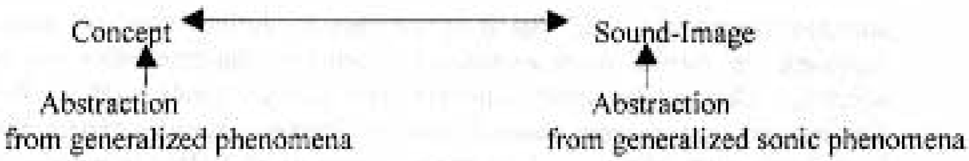
For example, in the hierarchy of concepts, the above figure, *green, yellow, red* and *triangle, circle, oblong* are primary concepts and color is a secondary concept, which is formed when we realize what the concepts green, yellow, red etc. have in common. Thus, the formation of the concepts *attribute* involves more stages of abstraction than the concepts of color, and shape. The terms 'higher-order' and 'lower-order' describe the same relationships as 'more abstract' and 'less abstract,' respectively.

Klausmeier explained concept as information of occurrences, things, or processes that enables the generalization of these occurrences, things, or processes. Thus, concept can be defined as "a general and comprehensive idea of common factors out of various ideas" or "mentally constructed common properties from many things, phenomena, and relations by subtracting (abstraction) and by abandoning (projection) non-common factors." A comprehensive definition of concept would be information that consist of common factors by eliminating non-common factors out of many things, phenomena, or processes.

Mentioned earlier, the term "concept" is a term that is used in mathematics education rather than in pure mathematics. Many mathematics educators and ducators explain "concept" similar to Skemp's classification. Skemp also explained ordinary concept and mathematical concept. Skemp's idea is consistent with Leslie Steffe and Paul Cobb, who have studied children's formation of number concepts for a long time, classified as "figurative concept" and "numerical concept" at the degree of abstraction.

Understanding of inclusive relations in mathematics enables students to make a deep mathematical understanding. For example, inclusion relation of figures (square<rectangle<trapezoid<quadrilateral) is hard for elementary school students to understand, but students can understand each concept clearly if they can understand the inclusion relations.

De Saussure explained the relationship between "concept" and "sound-image" as Figure 1. The interaction between concept and sound-image is bidirectional and associated each other. This association occurs in the abstractive level, which is different from the sensory-motor level. He argue that rich experience facilitate the association of concept and sound-image.



[Figure 1: The relationship of concept and sound-image]

As mentioned earlier, mathematical concepts are kinds of second-order concepts, which is different from first-order experience such as yellow, red, green, rectangular, oblong, hard, soft, and so forth.

5. A Model of Mathematical Concept Formation

In the 7th Korean curriculum, students should learn basic concepts through phenomena in daily lives and concrete examples and the basic concepts should be taught and learned from concrete to abstract sequence based on students' experiences and desires.

The learning of each concept should be occurred on the basis of concrete and individual activities because the straight forward definition of concepts are difficult for elementary school students to construct mathematical schemes. That is, because formal and logical learning is almost impossible for elementary school students at their developmental level, use of inductive reasoning that abstracts common properties from each specific fact is desirable in teaching mathematical concepts to the students.

It is different between a concept and some activities to explain the concept. For example, it is different between understanding commutative law and knowing the fact that the sum of numbers of beads are same regardless of left-to-right counting and left-to-right counting when 2 beads and 3 beads are on a table in a row. It is confusing to differentiate a concept and the name that is used to identify the concept.

Mathematics mainly deals with ideal objects that do not exist in our daily lives, whereas mathematics education in elementary level mostly deals with real objects that are available in our daily lives. In elementary level, we should teach mathematics with real objects and teach how to solve real world problems using mathematics. Therefore, we cannot teach students on ideal world beyond our daily lives.

In mathematics education, mathematical concepts are based on real situations. Thus, teachers should help students understand mathematical concepts by defining the concepts on the ground of real world. Students understand concepts not directly from real world, but from a model that comes from the real world. The model helps students understand abstract concepts. Figure 2 represents these processes.

[1] Real World

For a new direction of mathematics education, as Paul R. Trefton and Albert P. Shulte suggested as "mathematics can be learned from anything," we can change all elementary level of mathematics into situations in physical phenomena. Thus, to teach a mathematical concept, we should offer situations that are related to the concept and that are familiar to the students.

(i) Intuition

The importance of intuition has continuously been emphasized by many educators in various fields as well as by mathematics educators. However, there is a tendency that an intuitive attitude easily drives us to a too subjective knowledge.

On intuitive thought, Byung Lim Lyu explained that "as a recognizing activity," it is a thought to be able to figure out the whole even though it is not clear enough and to guide to direct to the process of theories from concrete situations and connect theories and concrete situations. Bruner argued that intuitive people invent or discover problems that logical people cannot do. His argument means that intuition should be the first step to creative aspects of mathematics. Creative mathematics should essentially connect to intuition.

(ii) Concrete Operational Activities

According to Piaget, elementary school children can understand by observing in real situations or by using concrete or semi-concrete manipulatives because they are in the concrete operational period. Trefton and Shulte argued that mathematics should be learned by doing. Hence, teachers should provide real world situations in which the students can observe directly from or concrete manipulatives with which students can play.

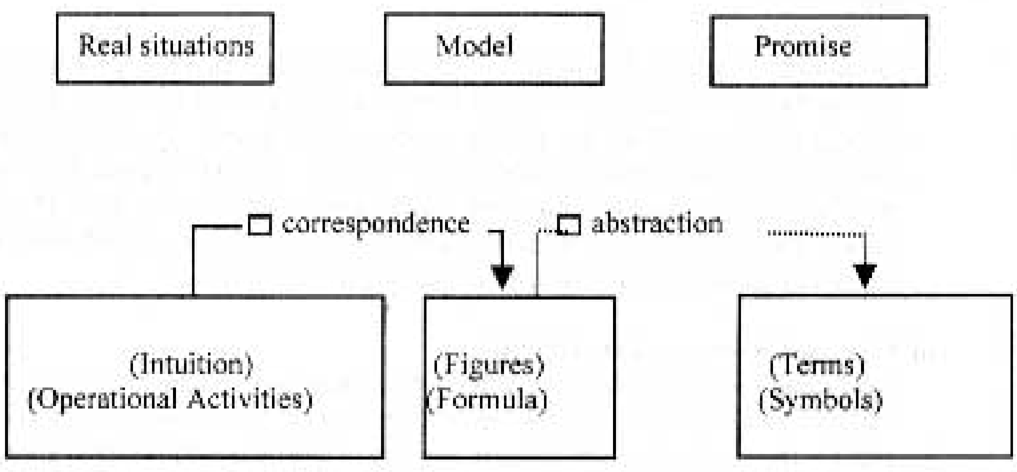
[2] **Model**

Mathematics is originally based on the real world. However, the real world itself is not the contents to teach in mathematics education. There are negative factors in light of time and pedagogical effects as well when we use real world situations in every mathematics lesson in the classroom. In some cases, it is not impossible to do that in the classroom. To compensate this defect, we should make models to correspond to real situations.

As a prior step to a theoretical definition, a model is a simple figure or a fact that corresponds to a real situation. The model helps students understand the properties of the concepts with ease. Compared with real situations, these figures and facts are sorts of semi-concrete manipulatives. Students can foster ability of abstraction by drawing figures that correspond to real situations in which the students had mathematical activities and by making formula that correspond to the semi-concrete objects such as figures.

[3] **Promise**

Mathematical concepts include processes and results rather than only results. In the process of obtaining mathematical concepts in mathematics education, one selects one perspective out of many perspectives to abstract definition of a concept. Definition is an outcome of formation of a mathematical concept in the field of mathematics education. More children-friendlier term "promise" replaces a rigorous mathematical definition. This promise consists of terms and symbols.



[Figure 2: The formation of mathematical concepts]

Students can develop ability to abstract concepts by deciding and using mathematical terms out of figures and formula and by representing with symbols correspond to the figures and formula. Teachers should select which part is suitable for the students in the circulation model depend on the students' mathematical levels.

6. The Practice of Teaching Mathematical Concept Formation

Division consists of measurement division and partitive division. Thus, division should be taught in two ways.

(1) Division as Measurement

□ Activities in the real world

□ Introduction in real lives

Students should learn mathematical basic concepts in the situations of students' lives. That is, teachers should use materials that are familiar to students' lives to teach the concept of division by measurement so that the students can recognize the meaning of it. Therefore, teachers can introduce division by measurement using candies that is familiar to students.

Bill want to make 2 packs of candies out of 6 candies to present to a friend. How many packs can you make?

□ Activities

The Korean curriculum emphasizes concrete operations and thinking processes and suggests that teaching and learning process should be from concrete to abstract. Therefore, for the activities of packing twos out of 6 candies, teachers suggest students take off 2 candies out of 6 as follows.

Let students do activity of taking off 2 candies out of 6.

□ Model

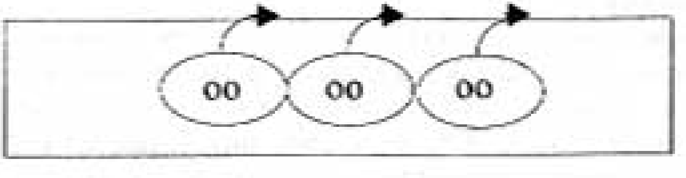
□ Drawing figures

The Korean curriculum recommends teaching and learning process of from simple concrete things to abstract things. Thus, teachers can let students draw semi-concrete figures corresponding to concrete situations. Students may draw the below semi-concrete figures corresponding to the candies as below.

○ ● □ □ □ □ ▫ ■ □ ▲ □ ▼

In addition, any remarks can be possible to be used by students to replace the concrete things, candies. Students can foster the ability of abstraction by drawing semi-concrete remarks as the correspondence to concrete things.

Students may draw figures like below to take off 2 candies out of 6 candies.



□ Writing formula

This is writing a formula from the semi-concrete figures according to the principles that keeps the sequences from concrete to abstract introduction. Students can foster the ability of abstraction by drawing semi-concrete remarks corresponding to concrete things. From the semi-concrete figures of taking off 2s out of 6 candies, students may make a formula as below.

6-2-2-2 = 0

□ Promise

□ Making promise

To represent concepts, students can use terms and symbols. Thus, teachers can let students represent with abstract terms and symbols out of figures and formulae. Terms and symbols are defined as follow.

□ Notice

□ The following concepts should be preceded to develop division by measurement.

Natural number 2, packing unit, subtraction, 0

To reinforce the concept of division by measurement, the following problems can be introduced.

Divide the following problems.

10÷2=□ 12÷3=□ 15÷5=□

- Draw figures to fit to the formulae 8÷2=4.
- Make a sentence to fit to the formulae 8÷2=4.

(2) Division as equal partition

□ Activities in the real world

□ Introduction in real life

Students should learn mathematical concepts in the situations of students' lives. Therefore, teachers can introduce division as equal partition using some pieces of bread that is familiar to students.

With 6 pieces of bread, Susan wants to share same amount of bread with two friends on two plates. How many pieces of bread can be put on each plate?

□ Activities

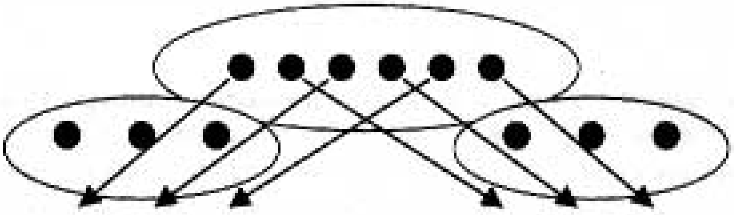
The Korean curriculum emphasizes concrete operations and thinking processes and suggests that teaching and learning process should be from concrete to abstract. Therefore, for the activities of putting same pieces of bread into two plates out of 6 pieces of bread, teachers suggest students put one piece of bread into each plate one by one in turn as follows.

Let students do the activity of putting 2 pieces of bread into two plates one by one in turns until the bread runs out.

□ Model

□ Drawing figures

The Korean curriculum recommends teaching and learning process from simple concrete things to abstract things. Thus, teachers can let students draw semi-concrete figures corresponding to concrete situations. Students can foster the ability of abstraction by drawing semi-concrete remarks corresponding to concrete things. Students may draw figures to put 6 pieces of bread into 2 plates at the same amount like below.



□ Writing formula

This is writing a formula from the semi-concrete figures according to the principles that keep the sequences from concrete to abstract introduction. Teachers can let students write on their own, but it is impossible to make a formula in this case. Thus, in the case of writing formulae can be omitted.

Definition

□ Making definition

To represent concepts, students can use terms and symbols. Thus, teachers can let students represent with abstract terms and symbols out of figures and formulae. Terms and symbols are defined as follow.

Divide 6 into two places with same amount of results 3 in each place. This is written as **6÷2=3** and read as 6 **divide 2 equals 3.**Formula like 6÷2=3 is called **division formula.**

□ Notice

□ The following concepts should be preceded to develop division by equal partition.

Natural number 2, Distribution one by one

□ To reinforce the concept of division by equal partition, the following problems can introduced.

Divide the following problems.

10÷2=□ 12÷3=□ 15÷5=□

· Draw figures to fit to the formulae 8÷2=4.

· Make a sentence to fit to the formulae 8÷2=4.

(3) Answering to the questions by students

□ Because the formula 3 ÷ 1/2 is a measurement division, students may write 3 ÷ 1/2 = 6. This means that 3- 1/2 –1/2- 1/2 –1/2- 1/2 –1/2 = 0. However, the formula $3 \div 1/2$ could not be thought as partitive division. For an example, we cannot divide 3 apples into 1/2 plates equally.

- ☐ We cannot subtract 1/2 from 4. Thus, the formula, $1/2 \div 4$ cannot be thought as measurement division. The formula $1/2 \div 4$ should be thought as a partitive division situation. There is 1/2 of an apple. If you want to divide this apple for 4 students equally, each student can eat 1/8 of an apple. Hence, the formulae can be $1/2 \div 4 = 1/8$.

  As shown above, division can be thought as measurement or equal partition situations. Thus, it is important for students to have a right concept of division.

7. **The Practice of Teaching to Develop Mathematical Concepts Using Pantomime**

It is effective to use pantomime to foster students' mathematical concepts correctly and develop logical and creative thinking in mathematics teaching. In the classroom, students draw figures and write formula as the correspondence to the teacher's pantomime. Students in groups communicate their own ideas and explain each other about their figures and formula. They may ask these questions: Why did you draw this way? Why did you write this formula? They can exchange their ideas and reflect their own ideas. This activity enhances students' communication and invigorates their various thinking, which leads to foster logical and creative thinking of students.

[1] Subtraction

Subtraction consists of subtraction by removal and subtraction by comparison. Thus, a teacher can perform two pantomimes to teach subtraction.

(1) Pantomime 1 (Subtraction by removal)

Teacher's Pantomime	Student's figure	Definition

(2) Pantomime 1 (Subtraction by comparison)

Teacher's Pantomime	Student's figure	Definition

[2] Division

Division consists of division by measurement and division by equal partition. Thus, a teacher can perform two pantomimes to teach division.

(1) Pantomime 1 (Division by measurement)

Teacher's Pantomime	Student's figure	Definition

(2) Pantomime 2 (Division by equal partition)

Teacher's Pantomime	Student's figure	Definition

[3] Computation of mixed operations

Let us think about teaching of Computation of mixed operations.

Student: Sir, what is the answer for the blank in $3 + 2 \times 4 = \square$?

Teacher: The answer is 11.

Student: Why is $3 - 2 \times 4 = 11$?

Teacher: If a computational problem has addition and multiplication, you should compute multiplication first and addition later. That is, you should compute 2 x 4, which is 8 and add 3 to 8, then you obtain 11.

Student: Sir, I think it is all right if we compute sequentially any complex computation problems. Why should we compute multiplication first and addition later? Is there any special intention of mathematician to confuse us?

Concerning these kinds of questions, a teacher can use pantomimes to help students understand procedural as well as conceptual knowledge.

Teacher's Pantomime	Student's figure	Definition

8. Specific Aspects in the Korean Textbooks

The Korean national elementary school mathematics textbooks consist of 'examining in real lives,' 'materials,' 'activities, ' 'promise,' 'methods to solve,' 'reviewing what already learned,' and 'exercises.' In addition, some chapters includes 'interesting games,' 'problem solving,' and 'challenge problems.'

The textbooks consist of several level of problems for which each level of students can emphasize on some specific sections of contents. One of the most prominent features of the contents is that each unit has a specific open question: "Do you think it is ___? Why do you think so?" About this question, students discuss each other in a free atmosphere. However, the goal of teaching mathematics is not the activities themselves, but the activities can be used as sources to teach mathematical concepts. Teacher's job is to provide effective materials and environmental in which students exchange their ideas.

The textbooks have a section of putting terms for mathematical concepts. In the textbooks, the section is called "promise." Instead of using a rigorous term "definition," the textbook use "promise," which is easier to use for students. Students have opportunity to consider how they informally name the assigned terms. This opportunity makes students have more meaningful concepts of the terms. Later, the teacher introduces a formal term for the mathematical concepts.

In the section of "methods," students have opportunities to think of various methods to solve problems. Among the methods, students decide which one is more convenient and simple way to solve the problems. In this section, teachers should assist students to understand why the formula came out and why students use formula. In the later sections, reviewing the previous contents, exercise, interesting games, challenging problems, and applying problems to the real lives are followed.

Korean mathematics textbooks include a type of workbooks as well. The workbooks include various types of problems that do not deal in the textbooks or more challenging problems. The workbooks can be used in the classrooms, but teachers are recommended to use the workbooks as supplementary materials to the textbooks.

9. Closing Remarks

The word "mathematical concept" is not the term that is in pure mathematics but in mathematics education. Thus, we should find the meaning of it in the methods of teaching mathematics. The lexical definition of it seems not to be difficult, but it is not easy to find concrete examples to teach mathematical concept.

Students can foster mathematical concepts by experiencing mathematical situations in their daily lives. They can use intuition and use manipulatives corresponding to the situations. Furthermore, they can develop mathematical concepts by drawing semi-concrete figures and writing formula, and then representing these with more abstract terms and symbols.

It is necessary for students to manipulate concrete objects, draw figures, and write formula to develop correct mathematical concepts. Students can develop the ability of abstraction while forming mathematical concepts. That is, students can develop the ability of abstraction by drawing figure and writing formula out of the corresponding to real situations, and by representing with terms and symbols with figures and formula.

For enhancing students' logical and creative thinking, pantomimes can be designed for students to think in various ways. Pantomime in the mathematics classrooms provide a wonderful opportunity for them to think traditional ways of strategies differently and to help them get the conceptual understanding of mathematical concepts. In addition, using pantomime, students can strengthen logical and creative thinking, thoughtful observation, meaningful representation and effective communication as well as expand their mathematical concepts accordingly.

Title : **Enhancing Children's Understanding of Mathematical Concepts Using Pantomine**

Presenter : Professor Jongsoo Bae, Seoul National University

Date & Time : 12 August 2003, 2.00 p.m. – 3.00 p.m.

1. Content of the Paper

1.1 The speaker demonstrated three examples of pantomimes to teach computation with mix operations in a more interesting manner. Pantomime helps students to develop interest and fun in mathematics, to think mathematics positively, to analyze and think logically and creatively, and to construct mathematics concepts correctly.

1.2 Video presentation of a lesson using pantomimes to teach elementary mathematics.

1.3 Brief overview of the Korean National Elementary Textbook that includes open-ended questions. Open-ended questions enhance students' mental representations, promote students' participation in a comfortable atmosphere and nurture literate citizens.

2. Discussion

2.1 *Professor Yoshihiko Hashimoto of Yokohama National University, Japan* asked how do we use pantomime to divide fractions with fractions?

Answer
Pantomimes can be best used in certain cases only.

2.2 *Question*
Norjoharuddeen Mohd. Nor of SEAMEO RECSAM Regional Centre for Education in Science and Mathematics, Penang, Malaysia asked if there was any rational in using pantomimes rather than acting and explaining at the same time.

Answer
Pantomime is a performance in which a performer expresses his or her intention by expressively bodily or facial movements. It is a gesture to stimulate students in thinking. Pantomime is a silent acting. It provides active learning and enhances students' communication and mathematical concepts.

Schemes of Children's Learning in Additive and Multiplicative Structures

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Abstract

Children used a variety of heuristic actions even before formal instruction, and these heuristics continue to be used for several years. Helping children to merge invented actions with the ones taught in school (often more abstract) requires careful development by the teacher. The goal sought was to identify children's transitional thinking schemes that bridge from addition, multiplication, proportional reasoning and develop them through. From the clinical interviews, the data suggests that mathematics instruction in the elementary school for these children has memorization and "mastery" of specific procedures as goals. While some children make sense of numbers and learn to compute using procedures prescribed by the teacher, many of them fail to understand what they are doing. This research identifies transitional thinking methods that bridge from addition, multiplication to proportional reasoning and call these methods "iterative multiplication." Students use these "iterative multiplicative" ways of thinking in each domain both to conceptualize the underlying problem situation and to carry out the numerical solution process.

Schemes of Children's Learning in Additive and Multiplicative Structures

Introduction

Studies have shown that students who score well on standardized tests often are unable to successfully use memorized facts and formulae in real-life application outside the classroom (Parmjit, 2000; Parmjit 2002;Yager, 1991). Resnick (1987) has commented that practical knowledge (common sense) and school knowledge are becoming mutually exclusive. This was echoed by Steffee (1994) :

> ***The current notion of school mathematics is based almost exclusively on formal mathematical procedures and concepts that, of their nature, are very remote from the conceptual world of the children who are to learn them.(P. 5)***

In our school climate, children's natural thinking "becomes gradually replaced by attempts at rote learning, with a disaster as a result" as indicated by Parmjit (2002) that the grades obtained in the national examination of Sijil Pelajaran Malaysia (SPM) for mathematics do not indicate their mathematical knowledge in problem solving. Often we hear in classroom with students increasingly say, "just tell me which formula to use, a way of saying, "don't ask me to think" and with teachers increasingly saying " we must 'cover' the syllabus, a way of saying, there is less time to think. Students and teachers are all victim, as we clamor for more mathematics results without realizing, that they may create less knowledge and more anxiety. It is crucial to stop just learning the rules. Two of the most important concepts children develop progressively throughout their mathematics education years are additivity and multiplicativity. Additivity is associated with situations that involve adding, joining, affixing, subtracting, separating and removing. Multiplicativity is associated with situations that involve duplicating, shrinking, stressing, sharing equally, multiplying, and dividing. This paper presents multiplicativity in terms of a multiplicative conceptual field (MCF), not as individual concepts. It is presented in terms of interrelations and dependencies within, between, and among multiplicative concepts. Additive thinking and corresponding addition strategies are often believed to inhibit understanding in traditionally multiplicative mathematical domains because many students persist in using additive operations and do not go beyond additive thinking to learn multiplicative approaches. Such persistence of insufficient additive strategies is well documented in multiplicative mathematical domains such as proportional reasoning and ratio understanding (e.g., Parmjit, 2001; Confrey & Harel, 1994; Greer, 1988). The common view is that multiplicative strategies must come to replace these inaccurate additive strategies in multiplicative problem situations.

Multiplicative reasoning plays an important role in the learning of mathematics and one could say that the lack of multiplicative reasoning in mathematics seems to be the critical factor in childrens learning. It is the foundation on which students construct a notion of ratio and proportion. Steffe (1988) has argued that the key to students' meaningful dealings with multiplication is the ability to iterate abstract composite units. This involves taking a set as a countable unit while maintaining the unit nature of its element. For example, if a child is ask, "If there are 6 groups of 3 blocks, how many blocks are there?" If the student can solve this problem by coordination of two number sequences, he or she has established an iterable composite unit. That is student, the student counts: 1 group is 3, 2 group is 6, 3 is 9, 4 is 12, 5 is 15 and 6 is 18.

Children used a variety of modeling heuristic actions even before formal instruction in additive and multiplicative structures, and these heuristics continue to be used for several years. Helping children to merge invented actions with the ones taught in school (often more abstract) requires careful development by the teacher. Children in school are often asked to voice or give their opinion, but at the end of the day they find out that their ideas are not being heard (Parmjit, 2001; Confrey, 1995). Confrey (1995) says that—"at one end of the spectrum by those whose mathematics was too weak to permit listening, and at the other end by those whose mathematical training was too effective to permit listening."(p. 6). Indications are that many concepts within the domain of additive and multiplicative structures are not well taught and therefore are not learned well. Determining what experience might be important to foster understanding requires a thorough analysis of the schema of children's thinking. In this paper, I seek to demonstrate that children's voice is an effective and accessible vehicle for making these challenges through discussion of the research I have conducted for the past few years. In light of this, the purpose of this paper is to identify children's reasoning schemas and develop them through.

Schema in Mathematics Learning

The concepts and experiences acquired by a student make up the knowledge that students possess. As new experiences occur, they are fitted into a person's existing mental structure. This is the Piagetian process of assimilation. Depending on the familiarity of the experiences and learning style of the learner, the experiences are received or rejected because of a person's mental structure or schema. The schema is a part of the mind used to build up the understanding of a topic. Thus to increase or alter what is already known, the schema takes in new ideas and fits them with what is already known. Understanding a concept means an appropriate schema has accommodated that concept. The view of what is meant by learning mathematics is, in Steffe's (1983) word, the "accommodation of schemas" in which existing mental schemas are modified in, or as a result of, their use. This language is, of course, metaphorical. We have no access to these schemas, but infer changes in cognition from changes that are observed in the behavior of the learner. The notion of "scheme" is useful. It allows us to discuss what we experience, namely that learning is difficult to recognize because so much of teaching produces certain changes in behavior without apparent "internal" change, that is change in the "scheme".

The idea of schema and how it functions provides a powerful tool for teaching mathematics. That a mental framework can be identified and developed means that mathematical relationships, patterns, and ideas can be understood rather than merely memorized; in the long run, children will have the ability to build up mathematical knowledge. When rules are memorized, children reach a point in their mathematical learning at which they are unable to remember the rules that are used to continue learning. Understanding has long since vanished. As mathematics is introduced, its understanding is predicted on children's having already developed appropriate early schemas. The implications are clear. Teachers should provide mathematical experiences in a form that will ensure that the mathematics is understood. Such a foundation provides a basis for all latter mathematical understanding.

I see vigorous pursuit of a child's voice as a means to articulate a broader set of perspectives on mathematics involving children's schemes in counting, multiplicative reasoning and how listening to children can lead to the development of powerful mathematical ideas in the teaching and learning of mathematics. The research reported here is not oriented towards conclusions about how to teach and introduce these concepts in children's early learning. However, it is intended to be conjectural, that is, to outline an approach to instruction based on a broad definition of mathematical knowledge that's seems to be worth trying.

Methodology

The data for this research focused on the ways students, from kindergarten, grade one, grade two and grade three, developed conceptual structures from their solution activity. Data collection was in the form of clinical interviews between the researcher and student where students solved a set of tasks related to additive and multiplicative structures. Two main advantages of utilizing clinical interviews were, first, allowing for intervention where students were encouraged to elaborate on their statements and judgments. This provides an opportunity to make valid inferences about students' covert intellectual processes (Opper, 1975). Second, this approach to gathering data provides for a continual interaction between inference and observation (Cobb, 1986b). Hence, the researcher continually tests about students' thinking and intervenes whenever the problem solving activity of the solver cannot be adequately explained by the model. The data presented for this presentation is based on the researchers past and on going research on children's learning in mathematics.

Data

These episodes were done separately with each child.

Kindergarten (February 2002)

This sessions were conducted separately with each child

R : **Displayed 4 cubes on the table. How many cubes are there?**

L : One, two, three, four…Four!
She tabbed at each cube as a perceptual factor in coordinating her counting schemes.

R : I add another 3 cubes. How many are there all together now?

L : One, two, three, four, …five, six, seven…Seven!

L counted from one and relied on the *counting all* of cubes (physical materials).
Children at this level rely on the ***count all*** and ***count on*** scheme in their scheme.

R: Displayed 4 cubes on the table. How many cubes are there?

J : Four (She answered immediately)

R : I add another 3 cubes. How many are there now?

J : five, six, seven…..Seven!

Instead of counting all cubes, she recognizes "4" represents all four cubes and ***counts on*** from there : " 5, 6, 7."

Episode February 2002 (Early Grade One)
This was a teaching experiment I did with the first grade students

R : **There are 5 cubes on the tables. I add another 3 cubes. How many are there in all?**
A : One, two, three, four, five, six, seven, eight.... (Counting each of them (one to one) using her fingers)
D : Five and three is eight.

R : I add another two more. How many are there in all?
A : One, two, three, four, five, six, seven, eight, nine, ten
D : Nine, ten.....ten!

Here, A used the count *one to one* heuristic while D used the *count on* method.
Episode April 2002 (grade one)

R : I have 11 marbles and gave 9 away. How many marbles do I have now?
Jas : Ten, eleven...two!
T : Ten, nine...two
A : Drew eleven marbles and cancelled nine of them.....two

Here, the three children used different heuristic to solve this problem.

Episode April 2002 (grade one)
Students were shown a card which stated 6 + 4 = __ + 5

Out of 25 students interviewed, 16 students of the class said ten was the answer! And another five of them said fifteen. Many of this children have a limited understanding of equality and the equal sign if they think 10 or 15 is the answer. In this session, three students were interviewed together.

R : Student showed a card which stated **6 + 4 = __ + 5**
L : because "six plus four is ten"
Jas : It is 5!
R : Why?
Jas : Because "six plus four is ten and the other side "five plus five is ten also"
R : ...but six plus four is ten
Jas : Both sides must be same

Both L and R seemed to agree with the explanation by J.
Many of the first grade children faced difficulties in constructing the meaning of "=". They see it as a symbol describing "compute it rather than a relationship sign. Children must be taught that equality is a relationship that expresses the idea that two mathematical expressions hold the same value.

Episode July 2002 (grade Two)

R : **showing a card : 42 + 9 =**
A : Nine plus two is eleven, carry one……four plus one is five…fifty one
S : Count by ten : 52 and go backward one…51
R : **72 – 28 =**
S : 38, 48, 58, 68, …69, 70, 71, 72,

She kept track of her iteration with the help of her fingers. She iterated by 10's in her counting scheme. She carried four fingers (tens) in her left hand and another four fingers (units) in her right hand.

K : Add 6 at the beginning and minus (it) out at the end; 72 + 6 is 78, 78 – 28 = 50, now minus (subtract) six …its 44.,

The following are the mistakes made by students in grade two, from the review of their books (Document analysis).

Type of mistakes commonly made by students (Grade two and three) :

23	56	44	50	55	69
- 14	- 27	- 35	- 38	+ 55	+ 31
11	31	11	28	10 10	9 10

Each of these examples, in different ways, show the result of imposing procedures, emphasizing counting and memorizing without opportunity to make sense of mathematics.

Conventional mathematics instruction in the elementary school has memorization and "mastery" of specific procedures as goals. While some students make sense of numbers and learn to compute using procedures prescribed by the teacher, many students fail to understand what they are doing, become frustrated, anxious and turn away from mathematics because it does not make sense to them. The examples above show why a shift from procedures to reasoning is essential

The following verbatim was with a second grade girl, which provided me with insights of multiplication reasoning in her thinking.
Episode 1 (July, 2002) (Grade two child)

R : How many two's are there in twelve?
S : 1,2; 3,4;5,6;7,8;9,10;11,12…..there are six.

Here, Sharan was coordinating units of units as she keeps track of the paired counting acts with her fingers. She counts by two's to twelve (2,4,6,8,10,12), keeping track of how many times she counted. This keeping track involves coordination of unit items at two levels called unit of unit, as shown below.

Unit	Unit
1	1,2
2	3,4
3	5,6
4	7,8
5	9,10
6	11,12

Here, her fingers help her in the coordination iteration process but I will say she iterated it additively rather than multiplicatively. This is because there was a pause in each finger for her to do the counting. At this stage of competency, her thinking was based on additive reasoning rather than multiplicative.

Episode 2 (August 2002)

R : There are 4 houses and in each house there are three cats. How many cats are there in all?

She used her fingers

S : Twelve cats!
R : Can you explain?
S : Three, six, nine, twelve, (utilizing her fingers)
R : What do you mean?
S : One house three cats, two house 6 cats, three house 9 cats and four house 12 cats!
R : Can you tell me the relationship between 3 and 12 in this problem?
S : Huh?
R : I mean, can you show or prove to me that there are twelve cats in the four houses?
S : Can I use the paper?
R : Yes

After drawing in the worksheet, she explained:

S : There are four houses (a drawing of four houses) and inside all these houses there are three cats…Three plus three plus three plus three is twelve!

At this stage of competency, she was able to iterate the 3 number pattern 3, 6, 9, 12 , fluently, and was able decompose the composite unit of 12 as 3+3+3+3 (as four three's). Here, she iterated the units multiplicatively and not additively. As mentioned earlier, to be considered additive, there would have been a pause in each finger in order for her to count mentally.

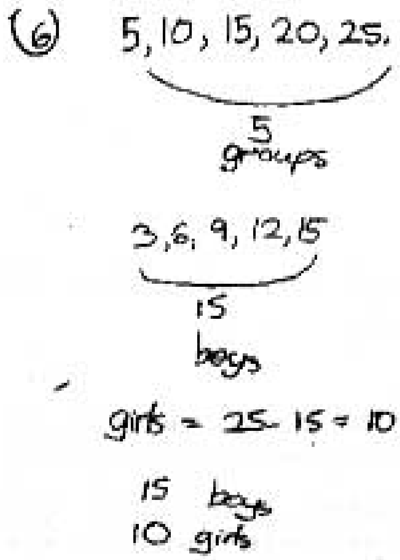
R : Ms. Lim put her students into groups of 5. In each group there were 3 boys. If she has 25 students, how many boys and how many girls does she have in her class?

Sharan : Are there five students in one group?

R : Yes.

After working on her work sheet, she gave her explanation referring to her worksheet as appended below.

S : There are five in a group…..five, ten, fifteen, twenty, twenty-five….there are 5 groups.



R : Ok.

S : But my question is how many boys and girls are there in her class?.

Sharan: Ok….three boys…..three, six, nine, twelve, fifteen….fifteen boys. There are fifteen boys.

R : How many girls?

S : twenty five minus fifteen….its … ten

R : How do you know that twenty five minus fifteen is 10, without using a paper?.

S : It is a number pattern…fifteen, twenty five, (it) is ten.

R : What do you mean fifteen twenty five is ten?

S : Fifteen, twenty five, thirty five, fourty five, ..it's a ten number pattern.

R : Good

Episode 4 (September, 2002)

R : To make coffee, Jenny needs exactly 3 cups of water to make 4 small cups of coffee. How many cups of coffee can she make with 12 cups of water.

S : Can I use the paper?

After working on her worksheet:

S : 16 cups

R : Can you please explain?

S : (Showing her worksheet) 3 cups you makes 4 , 6 you make 8 , 9 makes 12 and 12 makes 16.

3 water –	4 coffee
6 –	8
9 –	12
12 –	16

She coordinated the two number sequences of 3, 6, 9, 12 with 4, 8, 12, 16. This shows her ability to coordinate two number sequences.

December, 2002

(At this stage, Sharan has been introduced to the concept of multiplication in school). The following question was posed, similar to the one posed in September, 2002.

R : Mariam needs exactly 3 cups of water to make 4 small cups of coffee. How many cups of coffee can she make with 12 cups of water?.

After a while working on her worksheet, she responded:

S : 16 cups

R : How did you get that?

Showing me her worksheet, as appended below:

R : What does the arrow means?

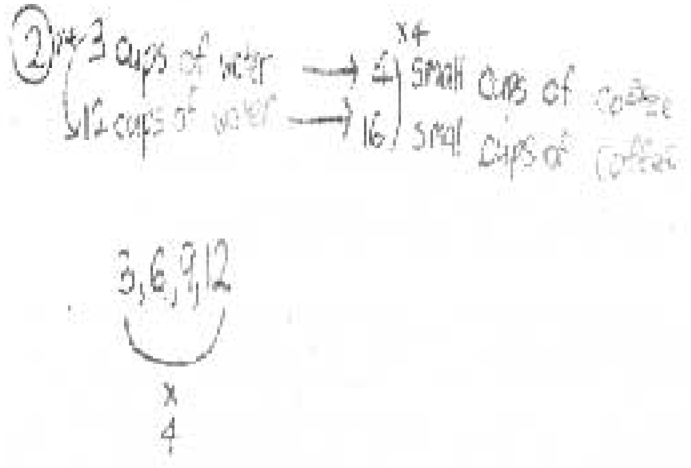
S : It means three cups of water to make four cups of coffee.

R : I see

R : Why did you times four here (pointing at her worksheet)?

S : Because in 3, 6, 9, 12 there are four....so times four!

R : Good



Sharan did not use the iteration process, as she had previously used, and instead utilized the relationship as a scalar function. I believe that she is beginning to conceptualize the iteration action of the composite unit to make sense of multiplicative facts in a problem.

In these problems, Sharan curtailed the iteration process by using known multiplication facts to aid in determining the total number of iterations. She then correctly multiplied the relevant composite units by that total. This curtailment required Sharan to sufficiently abstract the iteration action so that she could reflect on it and anticipate that the result of several iterations could be captured by a known multiplication fact. At this stage, it does indicate that she was beginning to represent the problem into symbolic representation. This reflects Sharan's ability to move from the iteration schemes to a more abstract level of understanding in multiplicative thinking. This level of competency from an early 3rd grade child represents Vergnaud (1983) representations of isomorphism of measures to illustrate the structure of multiplicative problems.

Conclusion and discussion

There are no easy recipes for helping all students learn or for helping all teachers become effective. Nevertheless, much is known about effective mathematics teaching, and this knowledge should guide professional judgment and activity. To be effective, teachers must know and understand deeply the mathematics abstraction of children when they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks. To illustrate the notion of abstraction in number sense from the data, consider the child who was asked to find the total of two collections of objects (eight objects and another five objects). Many young children will "count all" to find the total ("1, 2, 3, . . . , 11, 12, 13"), even once they are aware that there are eight objects in one set and five in the other. Other children will realise that by starting at 8 and counting on ("9, 10, 11, 12, 13"), they can solve the problem in an easier way. are therefore two important bench marks in children's developing understanding of Addition. To further illustrate let me describe one task from the interview. The interviewer asks "If you have 11 marbles, and you gave 9 away. How many would you have left?" Their reasoning schemas were : (This was based on all the interviews and some of them was not mentioned in the data because of space constraints).

Known fact

- Count down to (10, 9; so the answer is 2)
- Count up from (10, 11, so the answer is 2)
- Fingers used to keep track only (during counting down to or up from)
- Count back all (11, 10, 9, 8, 7, 6, 5, 4, 3, 2; so the answer is 2)
- Modelling all (shows 11 "things", then takes away 9 "things", leaving 2)

Teachers knowledge of children's schemas and having that as a repertoire of tools in their teaching will help young children in their learning. Thus children can capitalize on the strengths of different strategies and use each one for the problems for which its advantages are greatest. For example, for an easy addition problem such as 4+1, first graders are likely to retrieve the answer; for problems with large differences between the numbers, such as 2+9, they are likely to count from the larger number ("9,10,11"); for problems excluding both of these cases, such as 6+7, they are likely to count from one.

The data from the second/third grade child, Sharan, does suggest that multiplicative reasoning expressed by using iterative composite units (coordination of number sequences) can help children construct meaning for multiplication thinking in various multiplicative setting. As Sharan moved from additive to multiplicative reasoning with whole numbers, there are two significant related changes. There are changes in what the numbers are and changes in what the numbers are about. Steffee (1988) traces children's construction of numbers from the construction of single entities with singleton units to coordination with composite units that signals the onset of multiplication. It is not a trivial shift, because it represents a change in what counts as a number. The ability to use operations with composites (units of units) from the data presented seemed to involve three essential components. First, children needed to explicitly conceptualize the iteration action of the composite unit to make sense of multiplicative problem. Second, children needs to have sufficient understanding of the meaning of multiplication and division so that they can see their relevance in the iteration process. Third, and finally, children need to have sufficiently abstracted the iteration process so that they can reflect on it, then re-conceptualize it in terms of their knowledge of the multiplication and division operation.

Once it is recognized that children know multiple strategies and choose among them, the question arises: How do they construct such strategies in the first place? This question is answered through studies in which individual children who do not yet know a strategy are given prolonged experiences (weeks or months) in the subject matter; in this way, researchers can study how children construct their thinking in various stages. These I referred to as theorems in actions (Vergnaud, 1983) studies, meaning small-scale studies of the development of a concept. In this approach, one can identify when a new strategy is first used, which in turn allows examination of what the experience of discovery was like, what led to the discovery, and how the discovery was generalized beyond its initial use.

There are two purposes in doing research - for the construction of theory and informing practice. There is a wide agreement (Hiebert and Behr, 1988) that if research is to inform instruction, it is important to analyze mathematical structures and children's heuristic processes in light of the developmental precursors (or, sometimes prerequisites) to the knowledge needed to function competently in a domain. These precursors or cognitive building blocks have been called by many names: key cognitive processes (Hiebert and Wearne, 1988), key informal strategies (Hiebert and Behr, 1988), theorems in actions (Vergnaud, 1983). These are schema considered necessary for meaningful learning in concepts building and this schema on children's learning can only be inferred through qualitative research methodology. Although the emphasis in conducting research on teaching and learning of mathematics has increased over the years, the purposeful connection between the construction of theories and the informing practice on instruction remains an issue. Research is clearly needed to explore how knowledge of children's learning of mathematics can be applied to the design of instruction.

From the data provided in this paper, the idea of schema and how it functions provides a powerful tool for teaching mathematics. Children learn better when the teaching builds on their reasoning schemas and mathematics teaching should be built on children's reasoning schemas and expanded through the use of mathematical representations. In view of this, for meaningful and effective teaching to take place, teachers need to understand the different representation of schemas a child brings to the classroom, the relative strengths and weaknesses of each, and how they are related to one another (Wilson, Shulman, and Richert 1987). They need to know the ideas with which students often have difficulty and ways to help bridge common misunderstandings. Unless the teacher has this knowledge, her interpretations of and therefore the use that she can make of information on her students' reasoning to inform her own practice will be limited.

These statements and reasoning are ideal for effective changes to take place. However, in practicality, Malaysian school teachers are so burdened with their load of work in school en-composing both academic and non-academic work that there is little time for teachers to reflect or read to seek improvement. Research has shown that teacher have a great load of work in school. Secondly, the reading materials, especially the outcome of the research conducted (usually in tertiary education) does not reach the schools for teachers consumption. One wonders what research is for if the product of the research does not reach the clientele, the teachers. The purpose of doing research is for informing practice and the construction of theory. While the latter has taken place effectively, with researchers getting their pieces of research into journals, journals becoming thicker, researchers curriculum vitae increases, researchers gets promoted and so forth, what about the former (informing practice)?. These poor teachers, often accused as not reading more to seek improvement, are left behind in the latest development of seeking improvement in the teaching and learning of mathematics. The outcome is that they teach as they were taught. It must be acknowledged that in Malaysia, there has been a shift in focus from a transmission model of teaching to an emphasis on teaching for understanding. It is no longer a case of the student "working out what is in the teacher's head" but instead on teaching that aims to understand and build on what the student is thinking. It should be stated however that this shift is present in policy statements and curriculum documents more so than in the reality of classrooms, reflecting the challenge this poses for teachers.

Is elementary mathematics so simple that teaching it requires knowing of only the "math facts" and a handful of algorithms? The premise of this paper is that, quite to the contrary, this early content is rich in important ideas. It is during their elementary years that young children begin to lay down these habits of reasoning upon which later achievement in mathematics will crucially depend. We must focus on how children learn and understand mathematics and base instructional decisions on this knowledge. I believe that it is as crucial for the educational research community to identify bridging methods that will help children move from additive perspectives and methods to multiplicative perspectives and methods in ways that permit children to integrate all of their knowledge in these two areas. I believe that understanding effective implementations of such approaches will allow additive and multiplicative domains to become widely accessible to all children in their learning.

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