A Combined Abduction-Induction Strategy in Teaching Mathematics to Gifted Students-with-Computers through Dynamic Representation

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1. Introduction

Students today grow up in a culture, which mostly depends on visual representations, messages are delivered dynamically through pictures. Students are used to receiving information in a very active mode. While written forms of representation are still important, it is necessary to consider how mathematical ideas can be represented through a visually dynamic medium. This strategy itself may help some gifted students to investigate and explore interesting mathematical ideas in a new and different way.

Unfortunately, in most current enriched or accelerated programs for mathematically gifted, students often just solve more problems at faster rate without opportunities to develop their own mathematical ideas. As a result, gifted students rely mostly on procedural knowledge. They lack the opportunities to engage in challenging investigations, experimental environments, and higher-level mathematical thinking.

Gifted students in most schools now have access to computers in their classrooms, and an increasingly large percentage of these students have private computers at home. As the goals for technology education and the promises of educational change have grown, the hardware and the software used in both schools and homes have improved steadily (Holden, 1989). Students are provided opportunities to do research and apply complex thinking skills by working with real problems and computer simulations. Learning becomes fun and more challenging. Students are taught programming languages that aid them in making a computer become a real tool. All students in gifted and talented programs should be introduced to such computer applications and programming.

The use of multiple dynamic representations which promote students' exploration of mathematical ideas is relevant. Research indicates that positive gains in understanding of mathematical topics appear in cases when multiple modes of dynamic mathematical representations are used effectively. Multiple modes of representation improve transitions from concrete manipulation to abstract thinking, and provide a foundation for continued learning. This study investigates the effectiveness of experimental environments for gifted students-with-computers to explore mathematical ideas through dynamic multiple representations. The purpose of this talk is to share an combined abduction-induction strategy in teaching mathematics to gifted in experimental environments. Applying this strategy, students need to construct their own dynamic models to conduct their experimentation.

2. Dynamic Visual Representations

This section emphasizes some of the positive effects of visualizing in mathematical concept formation and to show how dynamic visual representations can be used to achieve more than just a basic, procedural and mechanical understanding of mathematical concepts.

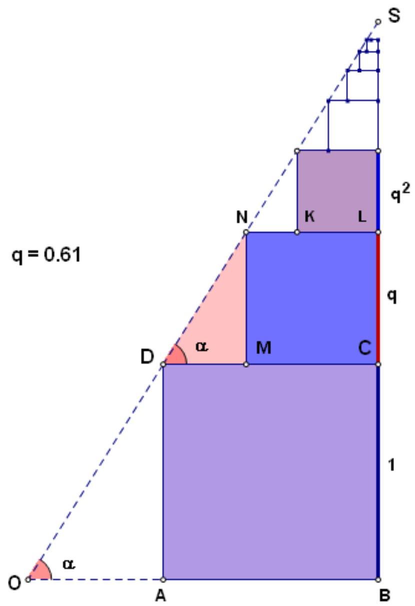
Arcavi (2003) proposed that:

> *Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.*

Example. Given a square *ABCD* with the side of 1 unit, on segment *CD* take a moving point *M*. Let *CM* = *q*; $0 \le q \le 1$. We assume $q \ne 0,1$. Construct square *MCLN* with the side *q*.

And then we continue to construct square with the side $KL = q^2$ and so on. Use a dynamic software to construct a geometric model for this situation and then find the sum of the infinite geometric series:

$$BS = 1 + q + q^2 + \cdots + q^n + \cdots$$



Investigation. From the picture, *DM* = 1 – *q*, so in the right triangle *DMN*, we have:

$$\tan\alpha = \frac{MN}{DM} = \frac{q}{1-q}$$

In right triangle *OAD*,

$$OA = \frac{1}{\tan\alpha} = \frac{1-q}{q}$$

We have: $OB = OA + AB = \frac{1-q}{q} + 1 = \frac{1}{q}$. In right triangle *OBS*,

$$BS = OB \times \tan\alpha = \frac{1}{q} \times \frac{q}{1-q} = \frac{1}{1-q}$$

In the dynamic models contructed by students, they can drag moving point *M* to change the value of *q* and observe the behaviour of the sum *BS*.

The computer is a rich source of visual and computational images that makes the exploration of mathematical conjectures possible. In this sense, the function of the software is paramount, providing the students with the opportunity to explore mathematical ideas, analyze examples and counter-examples, and then gain the necessary visual intuitions to attain powerful formal insights. However, it seems that, although visualization is recognized as relevant, the final objective continue to be the rigorous mathematical proof, as we already reviewed within communities of mathematicians.

A *visual approach* in the mathematical thinking process would be characterized by:

- Use of graphical information to solve mathematical questions that could also be approached algebraically.
- Difficulty in establishing algebraic interpretations of graphical solutions.

- No need to first run through the algebra, when graphical solutions are requested.
- Facility in formulating conjectures and refutations or giving explanations using graphical information.

In this case, the computer is used to verify conjectures, to calculate, and to decide questions that have visual information as a starting point.

3. Dynamic Multiple Representations

The use of multiple mathematical representations has been shown to increase students' capability in exploring of mathematical ideas. Nonetheless, while research indicates positive gains in student learning of mathematical topics, these gains appear in case when the multiple modes of mathematical representations are used effectively. Multiple representational software has been developed for computers at a speed which is difficult to keep up with. The importance of such an approach is it facilitates students' coordination of established mathematical representations such as tables, Cartesian graphs and algebraic expressions. Dynamic multiple representations could actually change the way students and teachers know about mathematics. We believe that computer technology and its different interfaces are changing the nature of the senses we use to communicate within a gifted student-with-computer.

Example: Dynamic multiple representations of fractions, percentages designed by The Geometer's Sketchpad. This model represents the relationship among fractions, percentages and bar chart at the same time on the screen with the values automatically changed when we drag *a*, *b*, or *c*.

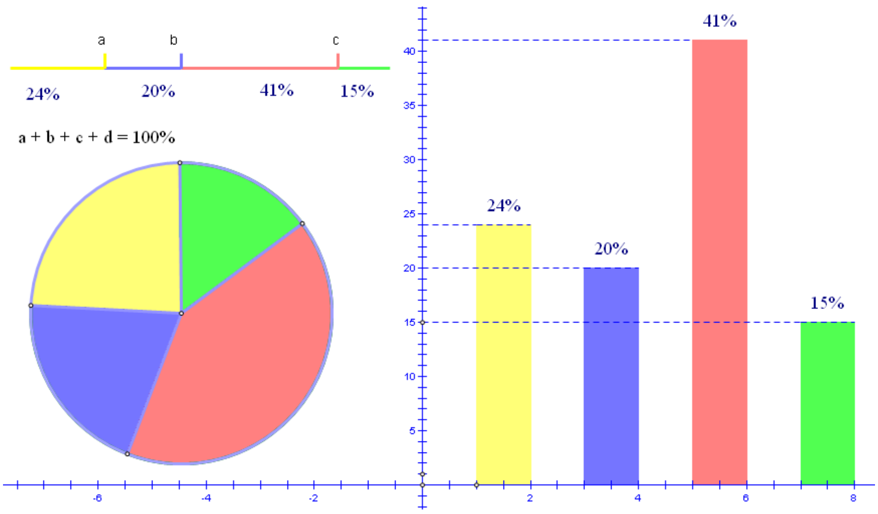


Figure 1. *Dynamic multiple representations of fractions, percentages and bar charts*

Manipulatives

The replication of physical manipulatives in the form of computer applications provides additional features and advantages over traditional manipulatives. Virtual manipulatives are advantageous in their capability to connect dynamic visual images with abstract symbols where physical manipulatives have limitations. Unlike physical manipulatives, virtual manipulatives use graphics, numbers, and words on the computer screen to connect the iconic with the symbolic mode and virtual manipulatives can record user interaction with the virtual manipulative and record such as movements and screen capture across time so the student or teacher can understand the false starts as well as the final submitted solution.

4. Gifted Students-with-Computers Exploring Mathematical Ideas in Experimental Environments

Experimentation associated with computers has a paramount role in mathematics education. Experimental mathematics has gained respectability in recent years, and that computers are partly responsible for this change. Mathematicians carry out experimental mathematics before the formulation of a conjecture they believe to be true and before the construction of a logic proof. Experimentation should be present more in schools for gifted students because computers are more available there, and experimentation is in resonance with collectives which involve computer technology. The use of geometrical software such as LOGO, Cabri or Sketchpad, the computer algebraic systems such as Maple, Derive, Mathematica, the graphing calculator or the so-called microworlds generate experimental environments that can be considered as laboratories where mathematical experiments are performed.

The experimental-with-computer approach

Educated trial and error, conjectures and refutations were elements of the logic of mathematical discovery. These elements characterized the students learning process in an experimental approach. More recently, we have started to think about this process as a way of thinking which is neither deduction nor induction but abduction, since the trials are very quickly no longer random. The logic of discovery is another way of characterizing abduction in the classical sense as described by C. S. Peirce. Abductive reasoning entails the study of facts and the search for a theory to explain them. It is the mode of inference dealing with potentiality:

- possible resemblance;
- possible evidence;
- possible rules leading to plausible explanations;
- possible diagnostic judgments;
- clues of some more general phenomenon.

Abduction

Reasoning which produces prediction and reasoning which explains observations are here considered cognitively different. The first is closely related to classical logical deduction, and the second is closely related to abductive reasoning and creative thinking. If Pierce uses abduction and deduction, Polya analogously introduces the idea of plausible reasoning in contrast to demonstrative reasoning (Polya, 1968). The relationship among abduction, induction and deduction was illustrated in Figure 2 of taxonomy of the inferential trivium.

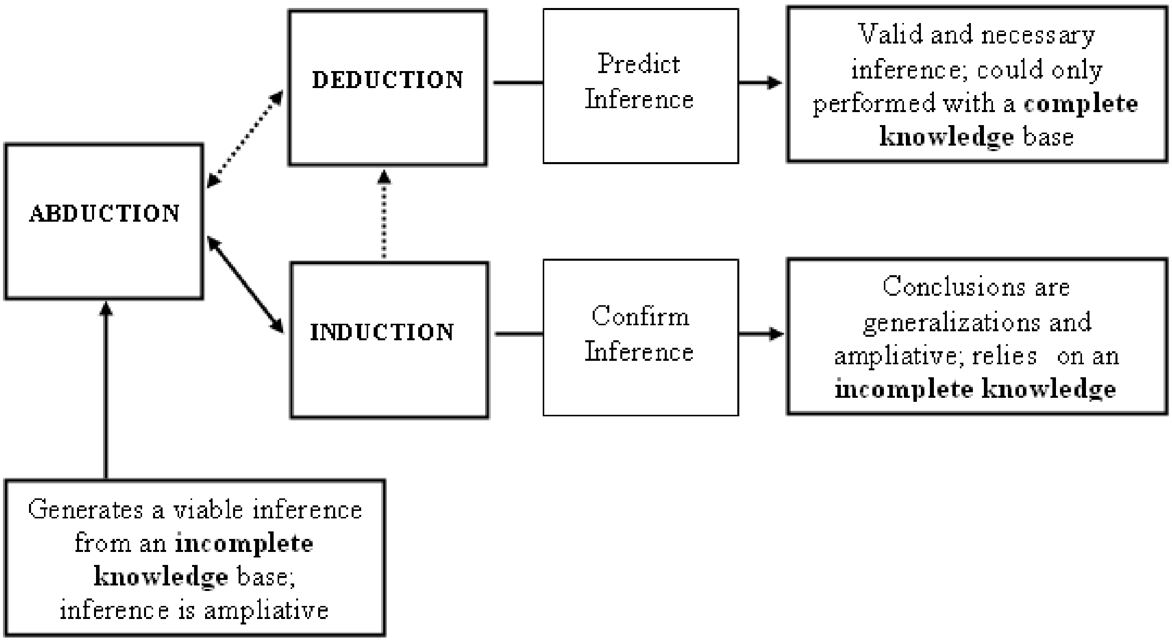


Figure 2. *Taxonomy of the inferential trivium* (Revised from Rivera and Becker, 2007).

For Abe (2003), Peircean abduction is another form of discovery or suggestive reasoning that "discovers new events" (p. 234) and yields explanations rather than predictions because they are not directly knowable. It is similar to induction insofar as both are concerned with discovery. However, it is distinguished from induction in that the latter "discovers tendencies that are not new events" (p. 234). Induction tests an abduced hypothesis through extensive experimentation and increased success on trials means increased confidence in the hypothesis.

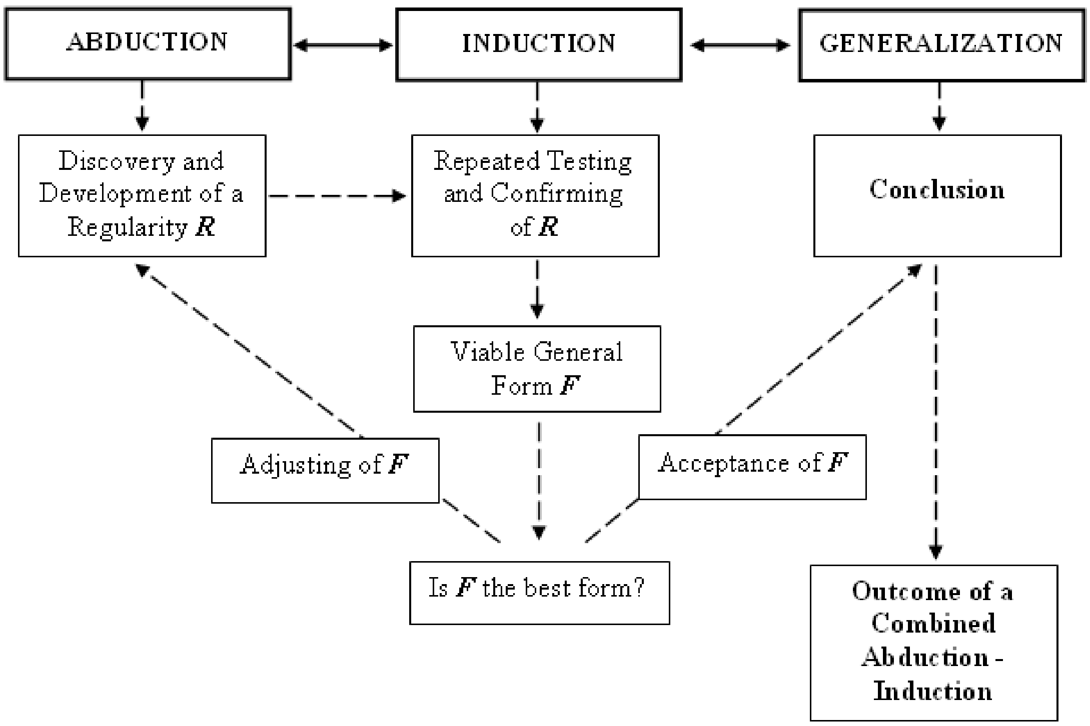


Figure 3. *Pattern Generalization Scheme* (Revised from Rivera and Becker, 2007).

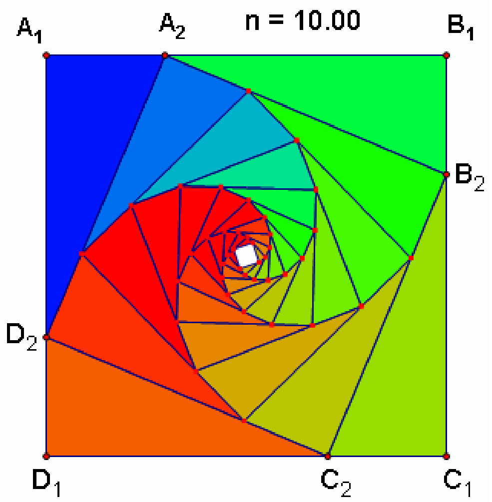
Figure 3 illustrates how the combined process materializes in a generalization activity from the beginning phase of noticing a regularity *R* in a few specific cases to the establishment of a general form *F* as a result of confirming it in several extensions of the pattern and then finally to the statement of a generalization (Rivera and Becker, 2007).

Example: Rotation, Dilation and Iteration

Given a square $A_1B_1C_1D_1$ with the side of 4 units. Construct square $A_2B_2C_2D_2$ as follows:

- Take A_2 arbitrary on A_1B_1. Calculate the ratio $\frac{A_1A_2}{A_1B_1} = k$. Then $0 < k < 1$.
- Construct B_2 that satisfies $B_1B_2 = A_1A_2$. Two points C_2 and D_2 are constructed similarly.
- From the square $A_2B_2C_2D_2$, construct square $A_3B_3C_3D_3$ as above and so on... $A_nB_nC_nD_n$,...

Let u_n be the side of the square $A_nB_nC_nD_n$. With a dynamic geometric software such as The Geometer's Sketchpad, students can construct their own model as shown in the below figure (Tran Vui, 2007b).



Students drag moving point A_2 to observe the change of the figure, there are many mathematical facts in this figure such as: rotation, dilation, iteration, geometric sequence, sum of infinite series… Students need to search for some "new theories" to explain these observed facts. Gifted students can conduct some experiments as follows:

- When $k = \frac{1}{4}$, find the formula of u_n.
- When *k* is arbitrary, find the formula of u_n in terms of *k*.
- When *k* is arbitrary, calculate the sum of the first *n* terms of the series.
- Define the ration and the dilation in this interation.

Visual representations in mathematics are more concrete and simpler than meanings mediated by verbal language but they are often also clearer and easier to understand. Due mainly to advances in computer software, pictures are today becoming a convenient vehicle for communicating ideas.

We can say that the experimental approach gains more power with the use of computers and thus, the experimental-with-computer approach provides:

- the possibility of testing a conjecture using a great number of examples and the chance of repeating the experiments, due to quick feedback given by computers;
- the chance of getting different types of representations of a given situation more easily;

- a way of learning mathematics that is resonant with modeling as a pedagogical approach.

5. A Combined Abduction-Induction Strategy in Teaching Mathematics to Gifted Students-with-Computers

The visual nature of multiple dynamic representations allows students to investigate algebraic and geometric properties. The dynamic visual representations on the screen generate experimental environments for students to manipulate with mathematical objects. Students have more opportunities to observe, test their guesses, make conjectures or counter-examples.

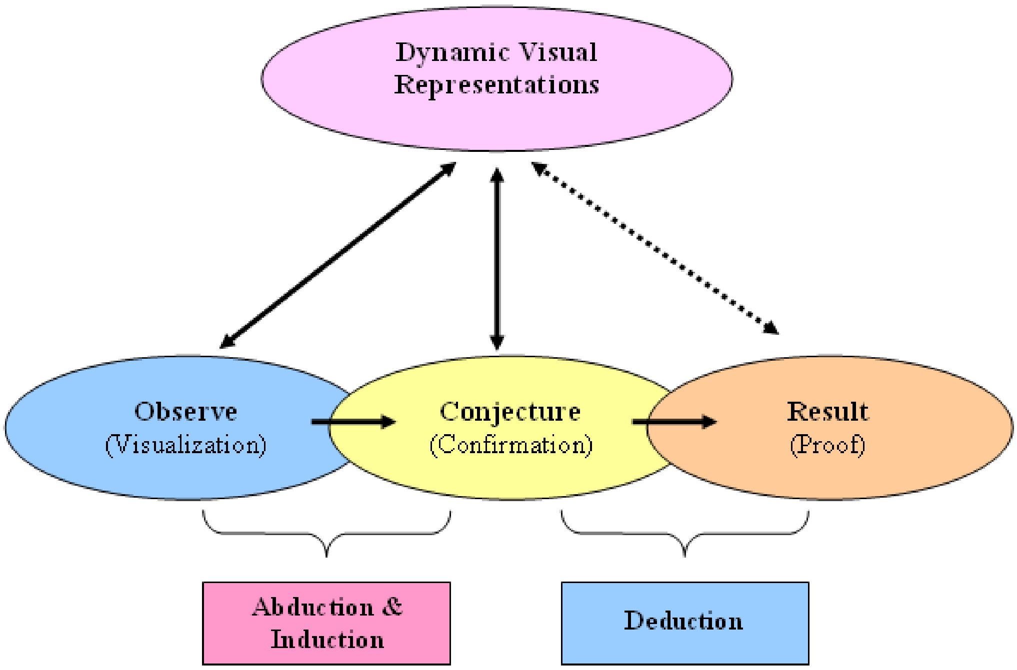


Figure 4. *A combined abductive- inductive strategy in teaching mathematics with dynamic visual representations*

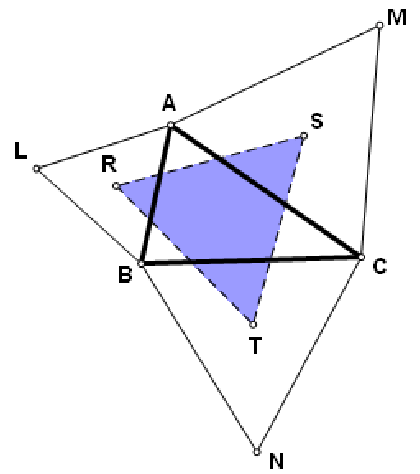
The interplay among dynamic representations, visualization, confirmation and proof can be illustrated by the model in Figure 4.

An investigation with The Geometer's Sketchpad

Example. *Take any generic triangle, and construct equilateral triangles on each side whose side lengths are the same as the length of each side of the original triangle. Surprise*: *the centers of the equilateral triangles form an equilateral triangle*!

1. Construct a triangle *ABC*, any triangle.
2. Construct equilateral triangles on the sides of the triangles.
3. Construct the centers of these triangles.
4. Connect the centers of these triangles. What is true?

Drag *A*, *B* or *C* to observe and collect the lengths of three sides *RS*, *ST*, and *TS*.



Use the "*tabulate*" option in "Measure Menu" to make a table of data. From these data students can predict a conjecture.

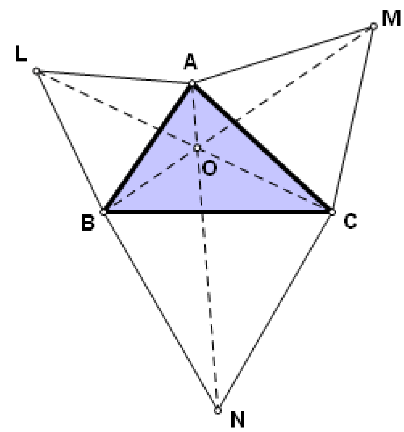
Conjecture 1. *The triangle RST is an equilateral triangle.*

Construct segments *AN*, *BM*, and *CL*.

From the figure as shown on the right, some conjectures can be made:

Conjecture 2. *Three segments AN, BM, and CL are equal.*

Conjecture 3. *Three segments LC, MB and NA intersect at a single point O.*

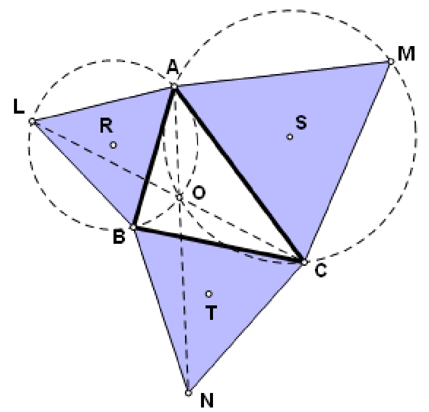


Conjecture 4. $\frac{Perimeter\ of\ (\Delta ABC)}{OR+OS+OT}=\sqrt{3}$

Conjecture 5. *The circumcircles of the three equilateral triangles ABL, BCN, and ACM have a common point.*

Conjecture 6. $\frac{OA+OB+OC}{ON+OM+OL}=\frac{1}{2}$.

Conjecture 7. *Three angles AOB, BOC, and AOC are equal to* 120 *degrees.*

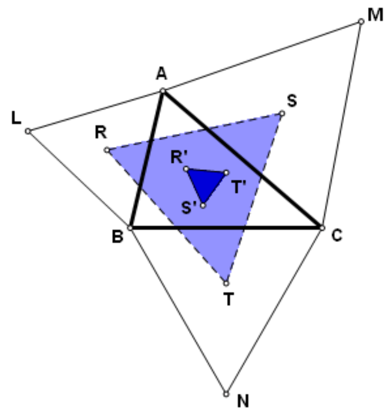


Conjecture 8. *Point O is the unique interior point to the triangle ABC that has the minimum total distance to three vertices A, B, and C.*

Let *R*', S' and *T*' be the image of *R* under the reflections about *AB*, *AC* and *BC* respectively.

Conjecture 9. *The* (*inner Napoleonic*) *triangle R'S'T' is an equilateral.*

Conjecture 10. *The difference in areas between the outer and inner Napoleonic triangles is the area of the original triangle ABC.*



A good dynamic geometric software gives gifted students more opportunities to construct their own models and observe many mathematical facts. Students can use "*Measure Menu*" to measure length, perimeter, angle, area, arc angle, arc length... to get more numerical data. From these data students make more conjectures based on their incomplete knowledge.

6. Training Global Teachers of the Gifted

There is a great need to provide opportunities for the education of teachers of gifted students that is applicable and accountable internationally. Using dynamic softwares to design effective dynamic multiple representations for exploring mathematical ideas require the learners a strong background in mathematics and also their skills in using the dynamic softwares. To prepare leading teachers of gifted know how to use dynamic representations we should:

- build up a knowledge bank of visual dynamic multiple representations or models for main mathematical ideas or challenging real-life investigations.
- set up virtual schools for mathematically gifted focused on dynamic multiple representations available for all gifted and accessible to international students.
- create more chances for teachers and students to explore mathematical ideas through online gifted education with an international and multicultural perspective.
- provide opportunities for teachers of the gifted to work internationally, and become global leading teachers themselves, are considerable in APEC region with the support of the internet.

The internet allows immediate and flexible access to vast resources, materials and leading mathematics teachers, and has changed the concept of knowledge from stable forms to fluid and fast changing. New roles for the teacher of the gifted beyond the classroom, faciliators encompass the monitoring, management and creative use of online formats in virtual environments accessible at any time, anywhere.

7. Conclusion

Do not force mathematically gifted students learn too much the knowldege invented by mathematicians long time ago. We should generate good educational environments for students to generate a viable inference from their incomplete knowledge base. Experimental environments based on dynamic multiple representations encourage gifted students to incorporate many different types of representations into their sense-making, the students will become more capable of solving mathematical problems and exploring underlying mathematical ideas. Dynamic mathematical softwares generate environments that can be considered as laboratories where mathematical experiments are performed. Trial and error, conjectures, refutations and generalizations are elements that characterize gifted students' work in these experiemental environments.

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